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Phase structure of the three-matrix chain model coupled to the magnetic field

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Abstract. We investigate the critical properties of the three-matrix chain model corresponding to the Blume-Emery-Griffiths model in a homogeneous magnetic field on a fluctuating lattice. The third-order critical line and the tricritical point disappear after the magnetic field is switched on. It is, however, found that another critical line exists as a result of naive perturbative expansion and the string susceptibility exponent is $-\frac{1}{3}$ on the line. It is also shown that the critical exponent of the magnetic moment β is $\frac{1}{6}$ which coincides with the KPZ solution.

1. Introduction

Recently the problem of the random matrix model has been investigated in the context of studying non-critical string theory and also two-dimensional statistical models coupled to gravity. One of the most interesting problems is whether the models describing minimal unitary series coupled to gravity [1] are given by an n-matrix model or not [2, 3]. It was found that the two-matrix model gives the correct critical exponents of the Ising model [4, 5] and the three-matrix model has been examined by the author and in [6].

In the previous paper [7] we found that the three-matrix chain model corresponds to the Blume-Emery-Griffiths (BEG) model with the lattice gas constant K being one in the unit of the spin exchange integral J on a fluctuating lattice. It was shown in [7] that in this model on the T-μ plane (T, μ are the temperature and the chemical potential respectively) the third-order critical line on the spin ordering extends to the tricritical point from the critical point of the Ising model. Along this line the string susceptibility exponent γ_{str} is $-\frac{1}{3}$, but at the end of this line, i.e. the tricritical point, it changes to $-\frac{1}{4}$. If one generalizes the chain matrix action adding another possible cross term, one can consider the other models for arbitrary value of K. For example, as will be discussed later, the cases of $K=0, 3, \infty$ correspond to the tricritical Ising model, three-state Potts model [8] and the Ising model [4] respectively. Our case of $K=1$ belongs to the intermediate region of K, and therefore it can be considered as the $(1,3)$-deformation of the conformal field theory [9] coupled to the gravity keeping the cosmological constant critical value.

This paper is concerned with the generalization of the previous paper to the case of a non-zero magnetic field. One of the advantages of the method of the matrix model is that it is exactly solvable. The model in the case of a regular lattice was considered using the method of the mean field approximation [10]. In the regular lattice calculation, the tricritical point disappears when a magnetic field exists and the new critical line appears for a given magnetic field in a certain interval of the chemical potential. The minimum value of the chemical potential on the line is at least greater than the value of the tricritical point. So it is interesting to examine whether the critical line exists for such a region of the chemical potential on a fluctuating lattice.

If the same structure as a regular lattice is also obtained for a non-zero magnetic field, it is demonstrated that the correspondence between matrix models and statistical models become clear. Hence it is important to work out the three-matrix model toward a generic case.

2. Formulation

To discuss the three-matrix chain model with the homogeneous magnetic field H let us consider the following path integral for three $N\times N$ Hermitan matrices u, v and t

$$Z_N(g)=\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}t\,\exp\left(-\mathrm{tr}\left\{u^2+v^2+t^2-2a(u+v)t+\frac{g}{N}(\mathrm{e}^{H}u^4+\mathrm{e}^{-H}v^4+\Delta t^4)\right\}\right) \tag{1}$$

where

$$a=\frac{c}{1+c}\qquad \Delta=\left(\frac{c}{1+c}\right)^2\mathrm{e}^{\mu}\qquad c=\mathrm{e}^{-2J}.$$

The free energy in the planar limit $N\to\infty$

$$F(g,a,\Delta)=\lim_{N\to\infty}\frac{1}{N^2}\ln\frac{Z_N(g)}{Z_N(0)} \tag{2}$$

is equivalent to the partition function of the BEG model

$$Z(g,J,\mu)=\sum_{m=1}^{\infty}\left(\frac{-gc^2}{4(1-c)^2}\right)^m\sum_{G^{(m)}}\sum_{\{\sigma,t\}}\mathrm{e}^{-\beta E} \tag{3}$$

as was shown in [7]. The Hamiltonian is given by

$$-\beta E=J\sum_{\langle i,j\rangle}G_{ij}^{(m)}\sigma_i t_i\sigma_j t_j+J\sum_{\langle i,j\rangle}G_{ij}^{(m)}t_it_j+H\sum_{i=1}^{m}\sigma_i t_i-\mu\sum_{i=1}^{m}(t_i-1) \tag{4}$$

where $\sigma_i=\pm1$, $t_i=0$, 1 and $G_{ij}^{(m)}$ is the adjacency matrix of the planar graph. As was discussed in [11], the integral of equation (1) can be rewritten by the $3N$ real integrals up to the irrelevant factors

$$Z_N(g)=\int\prod_{i=1}^{N}\mathrm{d}x_i\,\mathrm{d}y_i\,\mathrm{d}z_i\,W(x_i,y_i,z_i)\Delta_N(x)\Delta_N(y) \tag{5}$$

where

$$W(x,y,z)=\exp\left(-x^2-y^2-z^2+2a(x+y)z-\frac{1}{N}(g_1x^4+g_2y^4+g_3z^4)\right) \tag{6}$$

$$g_1=g\,\mathrm{e}^{H}\qquad g_2=g\,\mathrm{e}^{-H}\qquad g_3=g\Delta$$

and $\Delta_N(x)$ is the Vandermonde determinant. Let us introduce two orthogonal polynomials $P_n(x)$ and $Q_n(y)$ satisfying the orthogonality condition

$$\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\, W(x,y,z)P_n(x)Q_m(y)=h_n\delta_{nm} \tag{7}$$

and define the operators M_i $(i=1,2,3)$

$$\begin{aligned}(M_iO)(z)&=\int \mathrm{d}x\,\mathrm{e}^{-V_i(x;z)}O(x)\\ V_i(x;z)&=x^2+\frac{g_i}{N}x^4-2axz.\end{aligned} \tag{8}$$

It should be noted here that V_1 and V_2 are the counterparts of each other by the exchange $H\leftrightarrow -H$. Therefore the 'dual' transformation $H\leftrightarrow -H$ changes g_1, M_1 and P_n into g_2, M_2 and Q_n. We also introduce the variables R_i, S_i, $\alpha_{(k)ij}$ and their dual ones $\tilde{R}_i$, $\tilde{S}_i$, $\tilde{\alpha}_{(k)ij}$, $(i=1,\ldots,N;\ k=0,1,2)$ defined by the following equations:

$$\begin{aligned}xP_n(x)&=P_{n+1}(x)+R_nP_{n-1}(x)+S_nP_{n-3}(x)+\ldots\\ &=\sum_j \alpha_{(0)nj}P_j(x)\end{aligned} \tag{9a}$$

$$z(M_1P_n)(z)=\sum_j \alpha_{(1)nj}(M_1P_j)(z) \tag{9b}$$

$$z(M_3M_1P_n)(z)=\sum_j \alpha_{(2)nj}(M_3M_1P_j)(z) \tag{9c}$$

and their dual equations. Considering the integrals

$$I=\int \mathrm{d}x\,\mathrm{e}^{-V_3(x;z)}\frac{\mathrm{d}}{\mathrm{d}x}(M_1P_n)(x) \tag{10a}$$

$$J=\int \mathrm{d}z\,\exp\left(-z^2-\frac{g_3}{N}z^4\right)z(M_1P_n)(z)(M_2Q_m)(z) \tag{10b}$$

$$K=\int \mathrm{d}x\,\exp\left(-x^2-\frac{g_1}{N}\right)xP_n(x)(M_3M_2Q_m)(x) \tag{10c}$$

and their dual ones, one obtains the relations

$$\alpha_{(2)}=\frac{1}{a}\left(\alpha_{(1)}+\frac{2}{N}g_3\alpha_{(1)}^3\right)-\alpha_{(0)} \tag{11a}$$

$$\alpha_{(k)ij}h_j=\tilde{\alpha}_{(k)ji}h_i \qquad k=0,1,2 \tag{11b}$$

and their dual ones. Using the identities

$$\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,W(x,y,z)\frac{\mathrm{d}P_{n-1}}{\mathrm{d}x}Q_n(y)=0 \tag{12a}$$

$$\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,W(x,y,z)\frac{\mathrm{d}}{\mathrm{d}x}[P_n(x)-x^n]Q_{n-1}(y)=0 \tag{12b}$$

$$\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,W(x,y,z)\frac{\mathrm{d}P_{n-3}}{\mathrm{d}x}Q_n(y)=0 \tag{12c}$$

and taking the large-N limit

$$x \sim \frac{i}{N} \qquad \frac{h_i}{h_{i-1}} \sim Nf(x)$$
$$R_i \sim NR(x) \qquad S_i \sim N^2 S(x) \tag{13}$$
$$\tilde{R}_i \sim N\tilde{R}(x) \qquad \tilde{S}_i \sim N^2 \tilde{S}(x)$$

we finally get the following three equations

$$f(1+6g\,e^{H}R) = \tilde{R} + 6g\,e^{-H}(\tilde{R}^2+\tilde{S}) - \tfrac{1}{2}x \tag{14}$$

$$a^4(\tilde{R}+f) = a^2 f(1+6g\,e^{H}R) + 6\Delta g f^2[(1+6g\,e^{H}R)^2(1+6g\,e^{-H}\tilde{R}) + 2g\,e^{H}f(1+6g\,e^{-H}\tilde{R})^2 + 8(gf)^2(1+6g\,e^{H}R)] \tag{15}$$

$$a^4 S = 2gf^3[e^{-H}a^2 + \Delta\{(1+6g\,e^{-H}\tilde{R})^3 + 12g\,e^{-H}f(1+6g\,e^{H}R)(1+6g\,e^{-H}\tilde{R}) + 24g^3f^3\,e^{-H}\}] \tag{16}$$

and their three dual equations. The free energy in the planar limit can be calculated in terms of these equations and therefore these equations have much information about the solution. Actually the free energy and the magnetization are obtained in the following expressions:

$$F(z,a^2,\Delta,H) = \frac{1}{2}\ln\left(\frac{z}{g(z)}\right) - \frac{1}{g(z)}\int_0^z \frac{dt}{t}g(t) + \frac{1}{2g^2(z)}\int_0^z \frac{dt}{t}g^2(t) - \frac{1}{2}\ln\frac{3}{1-2a} + \frac{3}{4} \tag{17}$$

$$M = -\frac{1}{g(z_0)}\left.\frac{\partial g(z_0)}{\partial H}\right|_{H=0} \tag{18}$$

where $z = 6gf(1)/a^2$ and $g = g(z)$ is defined by

$$g(z) = \tfrac{1}{3}e^{H}\tilde{r}(\tilde{r}-1) - \tfrac{1}{3}a^2rz + \tfrac{1}{9}a^4z^3\tfrac{1}{9}\Delta a^2z^3(r^3\,e^{-H} + 2a^2r\tilde{r}z + \tfrac{1}{9}a^6z^3) \tag{19}$$

and

$$\Delta z^2[r^2\tilde{r} + \tfrac{2}{9}a^4z^2r + \tfrac{1}{3}a^2z\tilde{r}^2\,e^{H}] + (r-a^2)z + (1-\tilde{r})\,e^{H} = 0 \tag{20}$$

where $r = 1+6g\,e^{H}R$ and $\tilde{r} = 1+6g\,e^{-H}\tilde{R}$. z_0 expresses the critical surface of gravity determined by

$$\frac{\partial g(z)}{\partial z} = 0. \tag{21}$$

If one eliminates r and $\tilde{r}$ in the right-hand side of equation (19), the free energy is given by (17) as a function of z, a^2, Δ and H. However it is difficult, and so we consider the following perturbative expansions taking account of the duality

$$g(z) = \sum_{n=0}^{\infty} H^{2n} g_{2n}(z) \tag{22a}$$

$$r(z) = \sum_{n=0}^{\infty} H^{n} r_n(z) \tag{22b}$$

$$\tilde{r}(z) = \sum_{n=0}^{\infty} (-H)^{2n} r_n(z). \tag{22c}$$

Substituting equations (22) into (20) and expanding e^H into the power series of H, one can decide r_n order by order. Especially the all order calculations are possible in the case of $\Delta=0$ (Ising limit), i.e.

$$r_0=\frac{a^2z-1}{z-1} \qquad r_n=\frac{1}{n!}\frac{r_0-1}{z+1} \qquad n\geqslant 1. \tag{23}$$

Therefore from (19)

$$g_0(z)=\frac{1}{(1+c)^2}\left[\frac{c^2}{9}z^3+\frac{z}{3}\left(\frac{1}{(z-1)^2}-c^2\right)\right] \tag{24a}$$

$$g_{2n}(z)=\frac{1}{(1+c)^2}\frac{2z^2}{3(2n)!(1-z)^2(1+z)^2}. \tag{24b}$$

Equations (24) give the same expression as by Boulatov and Kazakov [5] up to the factor $1/(1+c)^2$. In the generic case it is difficult to calculate at all orders, however, it is enough to estimate up to the second power of H to obtain the magnetic moment M

$$M=-2H\left.\frac{g_2(z_0)}{g(z_0)}\right|_{H=0}. \tag{25}$$

3. Numerical results

The phase structures of our model are investigated numerically up to the order of H^2. The results for $H=0$ were reported in the [7]. In that case the third-order critical line exists in the region of $0\leqslant\Delta<\Delta^*$ ($\Delta^*=0.833$). And the tricritical point $\Delta=\Delta^*$ is just the fourth-order critical point. There is only the critical surface of gravity in the region $\Delta^*<\Delta$. On the other hand, after the magnetic field is switched on the phase structures are quite different from the former ones. The results are summarized in figure 1. In the region $0\leqslant\Delta<\Delta^*$ the third-order critical line disappears and only the critical surface of gravity survives. On this surface the string susceptibility exponent $\gamma_{\mathrm{str}}=-\frac{1}{2}$. However, the third-order critical line appears in the region $\Delta^*\leqslant\Delta<1$ and $\gamma_{\mathrm{str}}=-\frac{1}{3}$ on it. The gravity critical surface extends to the outer region $1\leqslant\Delta$.

The numerical results of the critical exponent β are as follows:

$$\begin{aligned} &\text{(a)} \qquad \beta=0.50 \qquad \text{for } \Delta=10^{-15}\\ &\text{(b)} \qquad \beta=0.49 \qquad \text{for } \Delta=0.7\\ &\text{(c)} \qquad \beta=0.17 \qquad \text{for } \Delta=\Delta^*. \end{aligned} \tag{26}$$

(a) is the case of the Ising limit and (b) the intermediate region which belongs to the universality class of the Ising model. (c) represents the tricritical point. The value of β and γ_{str} coincide with the KPZ [1] solutions for the tricritical Ising model ($\Delta_\sigma=\frac{1}{8}$, $\Delta_\varepsilon=\frac{1}{4}$ and $\Delta_{(1,1)}=-\frac{1}{4}$):

$$\begin{aligned} \beta&=\frac{\Delta_\sigma}{1-\Delta_\varepsilon}=1/6\\ \gamma_{\mathrm{str}}&=\Delta_{(1,1)}=-1/4. \end{aligned} \tag{27}$$

Therefore it is considered that (c) belongs to the universality class of the tricritical Ising model.

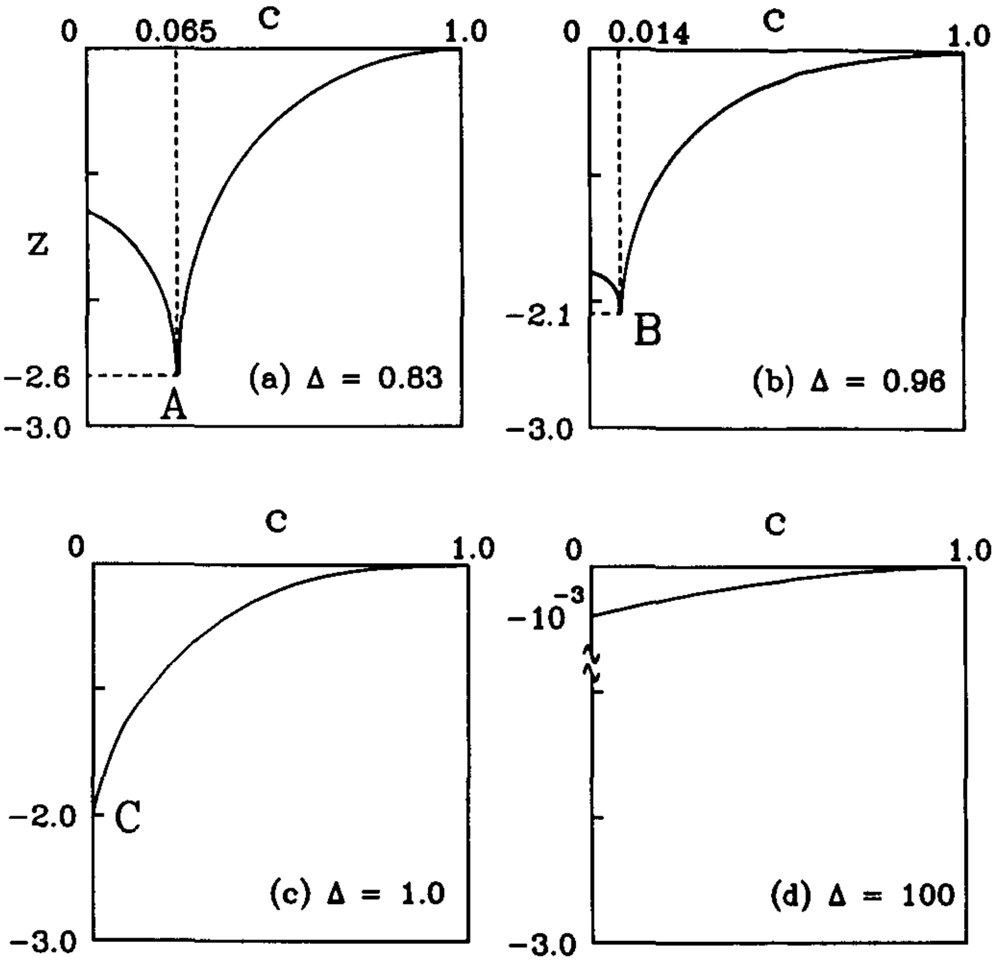

Figure 1. The global phase structures are shown when the magnetic field of $H=10^{-10}$ switches on. The separations are ignored. (a) The critical surface of gravity at $\Delta=\Delta^*=0.833$. A is the third-order critical point. The tricritical point disappears. (b) $\Delta=0.96$. B is the intermediate point between A and C. It is also a third-order critical point. (c) $\Delta=1$. The critical point disappears and only the critical surface of gravity exists. (d) $\Delta=100$.

4. Discussions

Finally some remarks are in order. In this paper the phase structures are studied up to the order of H^2. Near the third-order critical line the critical surface of gravity is separated into two pieces, a low-temperature piece and a high-temperature piece. Therefore one may suspect our results are incorrect. However, it should nevertheless be stressed that our model is a gravitational version of the model in [10] and certainly has the third-order critical line in the definite region of the chemical potential and that the case of $H=0$ does exactly correspond to the BEG model. Therefore it can be expected that all order calculations will lead to the structures without separation.

In consequence of the above consideration, it is conceivable that the m-critical Ising model is described by an m-matrix model extended to the cyclic chain. It is designed by the extended Dynkin diagram of su(m). This is correct for $m=2$. In the case of $m=3$ the extended three-matrix action, adding a uv-term for $H=0$,

$$A=u^2+v^2+t^2-2a(u+v)t-2buv+\frac{g}{N}(u^4+v^4+\Delta t^4) \tag{28}$$

gives the propagators

$$\langle u^2\rangle=\langle v^2\rangle=(1-a^2)/D \qquad \langle uv\rangle=(b+a^2)/D$$

$$\langle ut\rangle=\langle vt\rangle=a(1+b)/D \qquad \langle t^2\rangle=(1-b^2)/D$$

where $D=2[1-b^2-2(1+b)a^2]$. Comparing the propagators to the Boltzman weights for the Hamiltonian

$$-\beta E=J\sum_{\langle i,j\rangle} G_{ij}^{(m)}\sigma_i t_i\sigma_j t_j+K\sum_{\langle i,j\rangle} G_{ij}^{(m)} t_i t_j-\mu\sum_{i=1}^{m}(t_i-1) \tag{29}$$

J and K are parametrized by a and b,

$$\mathrm{e}^{-2J}=\frac{b+a^2}{1-a^2} \qquad \mathrm{e}^{-2K}=\frac{a^4(1+b)^2}{(1-b)^2(1-a^2)(b+a^2)}. \tag{30}$$

These equations are nothing but the projection from the J–K plane to the a–b plane. The line $K=0$ on the J–K plane is the tricritical Ising model. Hence it is expressed on the a–b plane by the curve

$$a^4(1+b)^2=(1-b)^2(1-a^2)(b+a^2).$$

The line $K=J$ which is the model of the previous paper is the line $b=0$. The line $K=\infty$ (Ising model) is $a=0$. $K=3J$ is projected on the line $a=b$. In this case $J=\mu/8$ and $K=3\mu/8$ when $\Delta=1$ and then the matrix action of equation (28) becomes Z_3-symmetric and the Boltzman weights for these values of J and K are exactly proportional to the Boltzman weights of three-state Potts model.

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